Systematic perturbation calculation of integrals with applications to physics

Paolo Amore,* Alfredo Aranda, and Ricardo Sáenz
Facultad de Ciencias, Universidad de Colima, Bernal Díaz del Castillo 340, Colima, Colima, Mexico

Francisco M. Fernández†
INIFTA (Conicet, UNLP), Diag. 113 y 64 S/N, Sucursal 4, Casilla de Correo 16, 1900 La Plata, Argentina
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In this paper we generalize and improve a method for calculating the period of a classical oscillator and other integrals of physical interest, which was recently developed by some of the authors. We derive analytical expressions that prove to be more accurate than those commonly found in the literature, and test the convergence of the series produced by the approach.

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I. INTRODUCTION

There is great interest in the development of new methods for the treatment of nonlinear problems [1–3]. Amore and Sáenz [4] have recently considered the problem of calculating the period of a classical oscillator with high precision. The method that they propose proves to be quite effective and applicable with limited effort to a large spectrum of problems. In fact, some of the problems that considered in Ref. [4] are textbook examples, for which such a method provides very accurate solutions; remarkably such solutions do not involve complicated transcendental functions but are expressed in terms of elementary functions. Amore and collaborators have recently discussed mainly two strategies for solving nonlinear problems. One of them is the direct treatment of the integral that gives the period of oscillation or any other relevant property of the system [4], and the other is based on an improved Lindstedt-Poincaré technique [5–7]. In both cases the authors resort to a sort of variational perturbation theory like that often used in quantum mechanics and other fields of theoretical physics to treat divergent series [8–10].

The purpose of the present paper is twofold: first, we want to generalize the method of [4] and to express it in a more systematic fashion; second, we want to extend the previous analysis to consider large orders and discuss the convergence of the expansions that we obtain. We investigate the systematic calculation of integrals with applications in various fields of theoretical physics, such as, for example, the period of nonlinear oscillations, the deflection of the light by the sun, and the precession of the perihelion of a planet orbiting around the sun. We try to provide simple though sufficiently accurate analytical formulas, and test the convergence of the series that give rise to them. In doing so, we compare our results with those in recent literature.

II. THE METHOD

Many physical problems reduce to the calculation of integrals of the kind

$$I = \int_{x_-}^{x_+} \frac{dx}{\sqrt{Q(x)}}, \tag{1}$$

where $Q(x)$ has simple zeros at x_- and x_+ and is positive in the interval $x_- < x < x_+$. Such integrals appear in many branches of classical mechanics as we will show shortly. In order to derive a simple analytical expression for the integral I we add and subtract a function $Q_0(x)$, which satisfies the same boundary conditions, and write

$$I = \int_{x_-}^{x_+} \frac{dx}{\sqrt{Q_0(x)}\sqrt{1+\Delta(x)}}, \tag{2}$$

where $\Delta(x) \equiv [Q(x) - Q_0(x)]/Q_0(x)$.

The method that we propose consists of expanding the integrand in powers of $\Delta(x)$ which leads to a series of the form

$$I = \sum_{n=0}^{\infty} I_n, \tag{3}$$

where

$$I_n = \binom{-1/2}{n} \int_{x_-}^{x_+} \frac{\Delta^n(x)}{\sqrt{Q_0(x)}} dx \tag{4}$$

and $\binom{a}{b} = \Gamma(a+1)[(\Gamma(b+1)\Gamma(a-b+1)]$ is the binomial coefficient. Notice that if we choose $Q_0(x)$ in such a way that $|\Delta(x)| < 1$ for all $x_- < x < x_+$ then the series (3) converges uniformly. The present method proves to be practical when we can obtain simple analytical expressions for a sufficiently large number of integrals in Eq. (4).

III. HARMONIC APPROXIMATION

According to what was said above about the function $Q(x)$ we can write

$$Q(x) = R(x)(x - x_-)(x_+ - x), \tag{5}$$

where $R(x) > 0$ in $x_- < x < x_+$. A simple suitable reference function $Q_0(x)$ for many physical problems is

$$Q_0(x) = \omega^2(x - x_-)(x_+ - x), \tag{6}$$

where ω is an adjustable parameter, so that

*Electronic address: paolo@cgic.ucol.mx
†Electronic address: fernande@quimica.unlp.edu.ar

$$\Delta(x) = \frac{R(x) - \omega^2}{\omega^2}. \tag{7}$$

In order to simplify the equations still further, we introduce the change of variable

$$x = \frac{x_+ + x_-}{2} + \frac{x_+ - x_-}{2} \cos\,\theta, \tag{8}$$

where $0 \leqslant \theta \leqslant \pi$. We can thus rewrite the integral (2) as

$$I = \frac{1}{\omega}\int_0^{\pi} \frac{d\theta}{\sqrt{1+\Delta}} \tag{9}$$

and the terms in the expansion (4) become

$$I_n = \frac{1}{\omega}\binom{-1/2}{n}\int_0^{\pi} \Delta^n \, d\theta. \tag{10}$$

Clearly, the integral I is independent of ω, but the partial sums

$$S_N = \sum_{n=0}^{N} I_n \tag{11}$$

will depend on that arbitrary parameter. It is therefore reasonable to require that the optimal value of ω should satisfy the principle of minimal sensitivity (PMS) [11] which states that

$$\frac{\partial S_N}{\partial \omega} = 0. \tag{12}$$

If we take into account the properties of the combinatorial numbers and that $\partial\Delta/\partial\omega = -(2/\omega)(1+\Delta)$, we can easily prove that

$$\frac{\partial S_N}{\partial \omega} = -\frac{2N+1}{\omega} I_N. \tag{13}$$

According to this equation, the PMS condition is equivalent to $I_N = 0$ which only takes place for odd values of N for real ω. For even values of N we may instead resort to the alternative PMS condition $\partial^2 S_N/\partial\omega^2 = 0$, but this additional requirement will prove unnecessary as we will see later on.

IV. ANHARMONIC OSCILLATORS

The periods of many anharmonic oscillators have been widely studied [1–3] and therefore they are suitable benchmarks for new approaches. Here we consider a particle of unit mass moving in a one-dimensional anharmonic potential $V(x)$ and calculate the period according to the well known expression

$$T = \int_{x_-}^{x_+} \frac{\sqrt{2}dx}{\sqrt{E - V(x)}}, \tag{14}$$

where $E = V(x_\pm)$ is the energy.

A. Duffing oscillator

A widely studied example is the Duffing oscillator [1], which corresponds to the potential $V(x) = \frac{1}{2}x^2 + (\mu/4)x^4$. In this case we choose $Q(x) = E - V(x) = (A^2 - x^2)\left[\frac{1}{2} + (\mu/4)(A^2 + x^2)\right]$, where $A > 0$ is the amplitude of the oscillations, and $Q_0 = \omega^2(A^2 - x^2)$. Notice that $T = \sqrt{2}I$ and that Δ takes a simple form:

$$\Delta = \frac{1}{\omega^2}\left[\frac{\mu}{4}(A^2 + x^2) - \omega^2 + \frac{1}{2}\right] = \frac{1}{\omega^2}\left[\frac{\mu A^2}{4}(1 + \cos^2\,\theta) - \omega^2 + \frac{1}{2}\right], \tag{15}$$

where $x = A\cos\,\theta$ as follows from Eq. (8) with $x_+ = -x_- = A$. Notice that the period depends only on $\rho = \mu A^2$.

The value of ω according to the PMS to first order

$$\omega_{PMS} = \sqrt{\frac{4+3\rho}{8}} \tag{16}$$

yields

$$\Delta_{PMS} = \frac{\rho}{4+3\rho}\cos 2\theta. \tag{17}$$

Notice that $|\Delta_{PMS}| < 1$ for all $\rho > -1$.

A most interesting feature of the PMS for this model is that $I_{2n+1} = 0$ for all $n = 0, 1, \ldots$, when $\omega = \omega_{PMS}$ given by Eq. (16). The calculation of the even terms is straightforward and yields the compact expression

$$T = \frac{4\pi}{\sqrt{4+3\rho}} \sum_{n=0}^{\infty} (-1)^n \binom{-1/2}{n}\binom{-1/2}{2n} \xi^{2n}, \quad \xi = \frac{\rho}{4+3\rho}. \tag{18}$$

If we choose $\omega^2 = 1 + \rho$ in Eq. (15) then Eq. (9) gives us a well-known exact expression for the period [1]

$$T_{exact} = \frac{4}{\sqrt{1+\rho}} \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - \kappa \sin^2\,\alpha}}, \tag{19}$$

where $\kappa = \rho/[2(1+\rho)]$. We thus obtain the alternative series expansion:

$$T = \frac{2\pi}{\sqrt{1+\rho}} \sum_{n=0}^{\infty} \binom{-1/2}{n}^2 \kappa^n. \tag{20}$$

When $\mu < 0$ the potential of the oscillator exhibits two barriers and the amplitude of the motion cannot be larger than $A_L = 1/\sqrt{-\mu}$. Consequently, the exact expression for the period is valid for $\rho > -1$. The present series (18) converges uniformly for $\rho > -1$ whereas the series in Eq. (20) does not converge for $-1 < \rho < -2/3$.

The exact solution $x(t)$ for the Duffing oscillator satisfies the virial theorem $\overline{\dot{x}^2} = \overline{x^2} + \mu\overline{x^4}$, where $\overline{f(x)}$ stands for the classical expectation value of $f(x(t))$. It is most interesting to note that the value of ω in $x_0(t) = A\cos\,(\omega t + \phi)$ that causes $x_0(t)$ to satisfy the virial theorem for the Duffing oscillator is $\sqrt{2}\omega_{PMS}$.

In a recent paper Pelster *et al.* [2] (PKS) calculated the leading term of the strong-coupling expansion for the frequency of the Duffing oscillator by means of a series produced by the Lindstedt-Poincaré method with an adjustable

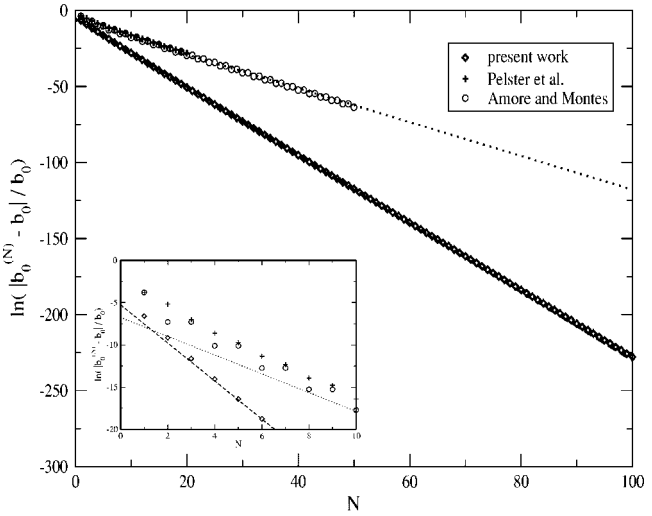

FIG. 1. Logarithmic plot of the error $|b_0^{(N)}-b_0|/b_0$ as a function of the order N. Diamonds and crosses correspond to present results and those of Pelster *et al.*, respectively. The dashed and dotted lines correspond to the linear fits of each data. The circles correspond to the results of [7].

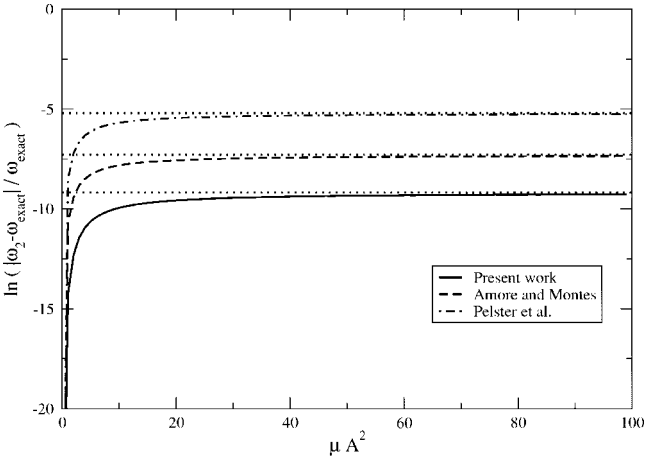

FIG. 2. Logarithmic plot of the error $|\omega_2-\omega_{exact}|/\omega_{exact}$ as a function of the order μA^2. The three curves correspond to the second order results for the present method, the LPLDE method of [5,7], and the method of Pelster *et al.*, respectively. The horizontal lines are the asymptotic values of the errors (see Fig. 1).

harmonic frequency. In the limit $\mu\to\infty$ they expanded the frequency for unit amplitude as

$$\omega=\sqrt{\mu}\left[b_0+\frac{b_1}{\mu}+\frac{b_2}{\mu^2}+\cdots\right]. \tag{21}$$

If $T^{(N)}$ is the Nth-partial sum for the series (18) we obtain the coefficient $b_0^{(N)}$ to order N as

$$b_0^{(N)}=\lim_{\mu\to\infty}\frac{2\pi}{\sqrt{\mu}T^{(N)}}=\frac{\sqrt{3}}{2\sum_{j=0}^{N}\left(\frac{-1}{9}\right)^j\binom{-1/2}{j}\binom{-1/2}{2j}}. \tag{22}$$

Figure 1 shows the logarithmic relative error calculated with this expression, with the values of $b_0^{(N)}$ given by Pelster *et al.* [2] and the results of Amore *et al.* [5,7]. We also show the linear fits for the first two sets of data.

Our series has by far the best rate of convergence. In fact, although all three series exhibit exponential convergence

$$\left|\frac{b_0^{(N)}-b_0}{b_0}\right|=e^{-\alpha-\beta N}, \tag{23}$$

the slope β of our linear fit is much greater: $\beta^{present}=\ln(9)\approx 2.1972$, compared to $\beta^{PKS}\approx 1.11$ [2]. Moreover, our expression (22), which is much simpler than the one of Pelster *et al.* [2], enables us to calculate the slope β exactly.

In Fig. 2 we compare the logarithmic relative error for the frequency calculated to second order with the three methods as a function of ρ. The results for the method of Pelster *et al.* are obtained by means of Eq. (42) of their paper [2].[1] In the limit $\rho\to\infty$ one recovers the asymptotic error of Fig. 1, corresponding to $N=2$. It is interesting to notice that all three methods yield an error which is always smaller than the asymptotic one.

B. Quadratic-sextic oscillator

We also consider the potential $V(x)=\frac{1}{2}x^2+(\mu/6)x^6$ because it has recently been treated by means of the combination of the methods of linearization and harmonic balance by Wu and Li [3] (WL). These authors carry out calculations of low order; their best approach is

$$T^{WL}=\frac{24\pi}{\sqrt{80+50\rho+\sqrt{4096+5120\rho+925\rho^2}}},\quad \rho=\mu A^4. \tag{24}$$

If we choose $\omega=\sqrt{(\rho+3)/6}$ in Eq. (9) with

$$\Delta=\frac{3+\rho-6\omega^2+\rho\cos(\theta)^2+\rho\cos(\theta)^4}{6\omega^2}, \tag{25}$$

we obtain an exact expression for the period:

$$T^{exact}=\frac{2\sqrt{3}}{\sqrt{\rho+3}}\int_0^{\pi}\frac{d\theta}{\sqrt{1+\frac{\rho}{\rho+3}(\cos^2\theta+\cos^4\theta)}}. \tag{26}$$

In this case the greatest amplitude for $\mu<0$ satisfies $\rho_L=\mu A_L^4=-1$.

The application of our method is similar to that for the Duffing oscillator discussed above. In order to keep our equations as simple as possible we use the optimal value of ω of first order to all orders:

$$\omega_{PMS}=\frac{\sqrt{5\rho+8}}{4}, \tag{27}$$

[1]Equation (42) of [2] contains a typo: the second term in the numerator should read $3g/2\omega_0^2$ instead of $3g/\omega_0^2$.

which leads to

$$\Delta_{PMS} = \frac{\rho}{3(5\rho+8)}[8\cos 2\theta + \cos 4\theta]. \tag{28}$$

The expressions thus obtained by our method through order three are not as accurate as the one of Wu and Li (24). At fourth order we have a formula that is not more complicated than Eq. (24) and is certainly more accurate for all values of ρ:

$$T^{(4)} = \frac{\sqrt{2}\pi(6097185\rho^4 + 37821440\rho^3 + 89272320\rho^2 + 94371840\rho + 3774873)}{2304(5\rho+8)^{9/2}}. \tag{29}$$

Notice, for example, that

$$\lim_{\rho\to\infty} \sqrt{\rho}T^{exact} = 8.413092631,$$

$$\lim_{\rho\to\infty} \sqrt{\rho}T^{WL} = 8.4081, \quad \lim_{\rho\to\infty} \sqrt{\rho}T^{(4)} = 8.41292$$

and that

$$T^{exact}(\rho = -0.9) = 10.93467798,$$

$$T^{WL}(\rho = -0.9) = 10.62, \quad T^{(4)}(\rho = -0.9) = 10.67.$$

Our method enables us to derive explicit equations at all orders. If we calculate the integrals in Eq. (10) exactly, we obtain

$$T = \sum_{n=0}^{\infty} \frac{4\pi}{\sqrt{5\rho+8}} \binom{-1/2}{n} \left[\frac{\rho}{3(5\rho+8)}\right]^n \mathcal{J}_n,$$

where

$$\mathcal{J}_n \equiv \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{n-k_1} 2^{3k_1-n} \binom{n}{k_1}\binom{k_1}{k_2}\binom{n-k_1}{k_3} \delta_{k1,2n-2k_2-4k_3}.$$

In Fig. 3 we plot the logarithmic relative error of the period for $\rho\to\infty$ as a function of the order of approximation. Once again we see that our expansion converges exponentially. Indeed, the error decreases as $e^{-\alpha-\beta N}$, where it is not hard to see that $\beta = \ln(5/3) \approx 0.5108$.

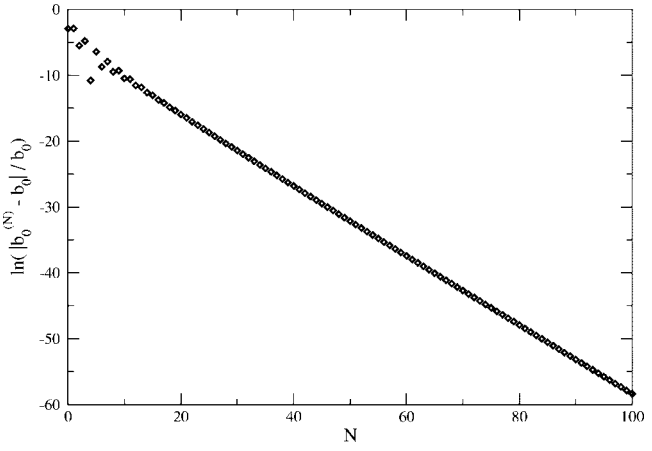

FIG. 3. Logarithmic plot of the error $|b_0^{(N)} - b_0|/b_0$ as a function of the order N for the quadratic sextic oscillator.

C. Other parity-invariant anharmonic oscillators

The anharmonic oscillators with even potentials of the form

$$V(x) = \frac{x^2}{2} + \frac{\mu x^{2K}}{2K}, \quad K = 2,3,\ldots \tag{30}$$

exhibit many features in common. For example, the period T depends on the parameter $\rho = \mu A^{2K-2}$ and satisfies a small-ρ expansion

$$T = \sum_{j=0}^{\infty} a_j \rho^j \tag{31}$$

and a large-ρ expansion

$$T = \frac{1}{\sqrt{\rho}} \sum_{j=0}^{\infty} c_j \rho^{-j}. \tag{32}$$

The optimal value of ω given by our method is of the form

$$\omega^2 = \frac{1+\kappa\rho}{2}, \tag{33}$$

where κ is the same to all orders for $K=2$ as we have already discussed above. It changes slowly with N for the sextic oscillator and more rapidly for greater values of K. In order to make the discussion simpler we just consider the convergence of the series for the most difficult case given by

$$c_0 = \lim_{\rho\to\infty} \sqrt{\rho}T. \tag{34}$$

For $K=5$ the value of κ given by the PMS of first order does not give us a convergent series for c_0. The reason is that the maximum value of $|\Delta|$ is greater than 1 under such condition. Values of κ given by the PMS of higher order correct this problem.

Let R_M and R_m be the maximum and minimum values of $R(x)$ $[R_M \geq R(x) \geq R_m > 0$ for all $x_- < x < x_+]$ and Δ_M and Δ_m the corresponding values of Δ. If we choose

$$\omega_b^2 = \frac{R_M + R_m}{2}, \tag{35}$$

then we have

$$\Delta(x) = \frac{2R(x) - R_M - R_m}{R_M + R_m} \tag{36}$$

and

$$0 < \Delta_M = \frac{R_M - R_m}{R_M + R_m} = -\Delta_m < 1. \tag{37}$$

Notice that the value of ω (35) that balances the maximum and minimum values of Δ makes $|\Delta(x)| < 1$ for all values of the coordinate.

Numerical calculation suggests that the values of κ given by the PMS at increasingly greater order approaches $\kappa_b = (2\omega_b^2 - 1)/\rho$. We find that κ_b is suitable for high-order calculations but it is not necessarily the most convenient one for the derivation of simple and accurate analytical expressions like those obtained from the PMS of low order.

Plots of $c_0^{(2N+1)}$ vs κ obtained by partial sums of order $2N+1$ show that the PMS always give the best approach to the exact value at each order. Plots of $\ln|(c_0^{(2N+1)} - c_0)/c_0|$ vs N suggest that our partial sums with κ_b converge exponentially for all anharmonic oscillators (30).

For negative values of μ there is periodic motion provided that $\rho > -1$. Plots of $\ln|(T^{(N)} - T)/T|$ ($\rho = -0.9$) vs N for anharmonic oscillators with $K = 3, 4, 5$ show two almost straight lines, one for even N lying always lower than the corresponding one for odd N. In other words, the partial sums with even N and odd N converge exponentially to the actual value of $T(\rho = -0.9)$.

D. Quadratic cubic oscillator

The treatment of noneven potentials is slightly different. We illustrate the procedure by means of the simplest such oscillator:

$$V(x) = \frac{x^2}{2} + \mu\frac{x^3}{3}. \tag{38}$$

The factorization of $Q(x) = E - V(x)$ leads to

$$Q(x) = (x - x_-)(x_+ - x)(b_0 + b_1 x), \tag{39}$$

where

$$b_0 = -\frac{x_+ x_-}{2(x_+^2 + x_+ x_- + x_-^2)}, \quad b_1 = -\frac{x_+ + x_-}{2(x_+^2 + x_+ x_- + x_-^2)}, \tag{40}$$

and

$$\mu = -\frac{3}{2}\frac{x_- + x_+}{x_+^2 + x_+ x_- + x_+^2}. \tag{41}$$

A straightforward calculation based on the change of variable (8) shows that the PMS at first order yields

$$\omega_{PMS} = \frac{\sqrt{(-x_+^2 - 4x_+ x_- - x_-^2)}}{2\sqrt{x_+^2 + x_+ x_- + x_-^2}} \tag{42}$$

and

$$\Delta = \frac{(x_+^2 - x_-^2)\cos\theta}{x_+^2 + 4x_+ x_- + x_-^2}. \tag{43}$$

It is clear that all the perturbation corrections of odd order vanish when $\omega = \omega_{PMS}$ and we obtain the same series as in the case of the Duffing oscillator:

$$T = \frac{\sqrt{2}\pi}{\omega}\sum_{j=0}^{\infty}(-1)^j \binom{-1/2}{j}\binom{-1/2}{2j}\xi^{2j}, \quad \xi = \frac{(x_+^2 - x_-^2)}{x_+^2 + 4x_+ x_- + x_-^2}. \tag{44}$$

The potential-energy function exhibits a maximum $V(x_M) = 1/(6\mu^2)$ at $x_M = -1/\mu$; therefore, there will be periodic motion for all values of energy satisfying

$$E < \frac{1}{6\mu^2} = \frac{2(x_+^2 + x_+ x_- + x_-^2)^2}{27(x_+ + x_-)^2}. \tag{45}$$

In this case we obtain the exact solution by means of the integral

$$T = \frac{\sqrt{2}}{\omega}\int_0^{\pi}\frac{d\theta}{\sqrt{1 + \xi\cos\theta}} \tag{46}$$

that yields the series (44) on expansion in a Taylor series about $\xi = 0$.

An interesting application of the main equations for the quadratic-cubic anharmonic oscillator is the precession of the perihelion of a planet. As shown in a previous paper by some of the authors [4] the expression for the angular precession $\Delta\phi$ that appears in most textbooks can be rewritten as

$$\Delta\phi = 2\int_{z_-}^{z_+}\frac{dz}{\sqrt{(z_+ - z)(z - z_-)[1 - 2GM(z + z_- + z_+)]}} - 2\pi, \tag{47}$$

where $z = 1/r$, $z_\pm = 1/r_\pm$ and $r_- < r_+$ are the shortest (perihelia) and largest (aphelia) distances from the sun. Here G is the gravitational constant and M is a mass. Taking into account the expressions for the quartic-cubic oscillator derived above we have

$$\omega = \sqrt{1 - 3GM(z_+ + z_-)} = \sqrt{1 - \frac{6GM}{L}}, \tag{48}$$

where the semilatus rectum L is given by $1/L = (z_+ + z_-)/2$. We also obtain

$$\Delta = \xi\cos\theta, \quad \xi = \frac{GM(z_+ - z_-)}{3GM(z_+ + z_-) - 1} = \frac{GM\sqrt{\dfrac{a - L}{a}}}{6GM - L}, \tag{49}$$

where $a = (1/z_+ + 1/z_-)/2$ is the semimajor axis of the ellipse. We thus obtain the series

$$\Delta\phi = 2\pi\left[\frac{1}{\omega}\sum_{j=0}^{\infty}(-1)^j\binom{-1/2}{j}\binom{-1/2}{2j}\xi^{2j} - 1\right] \tag{50}$$

that is a generalization to all orders of an expression of second order developed by Amore and Sáenz [4].

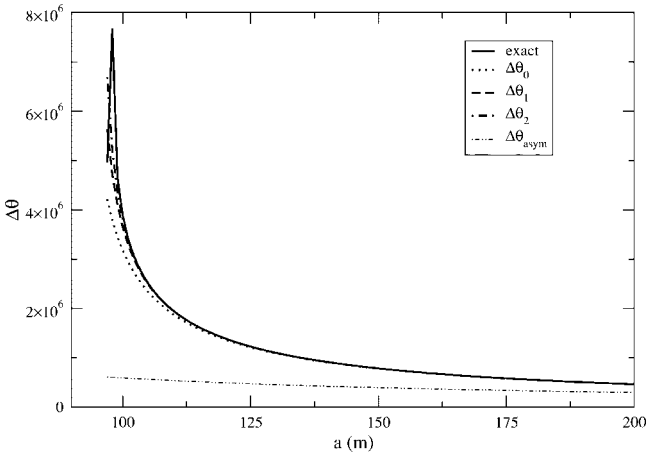

FIG. 4. Precession of the orbit of a planet (in seconds of degree) calculated assuming $M=1.97\times10^{30}$ kg, $G/c^2=7.425\times10^{-30}$ m/kg, and $\epsilon=0.2506$ as a function of a (i.e., the average of perihelion and aphelion) for values close to a black hole. The solid line is the exact result; the thin line corresponds to the asymptotic formula; the other lines correspond to the different orders to which our approximation has been applied.

In Fig. 4 we plot the angular precession of the orbit of a planet obtained using the exact formula and the approximations given by partial sums of our series (50) for values of a very close to the Schwartzchild radius. In Fig. 5 we plot the logarithm of the relative error over the angular precession. We notice that our approximation is able to reproduce the exact result very accurately, even for very small a. However, our series does not reproduce the singular point of the integral (47) exactly, because this singularity is due to the appearance of the third root of the cubic polynomial inside the integration interval. In other words, the factor function $R(\theta)$ is negative for some values of θ when a is smaller than the critical value $a_c=97.9173$. In spite of this limitation, the partial sums of our series approach the exact value of the integral as we add more terms. We can improve our results by splitting the integral into two parts at the new zero of $Q(z)$

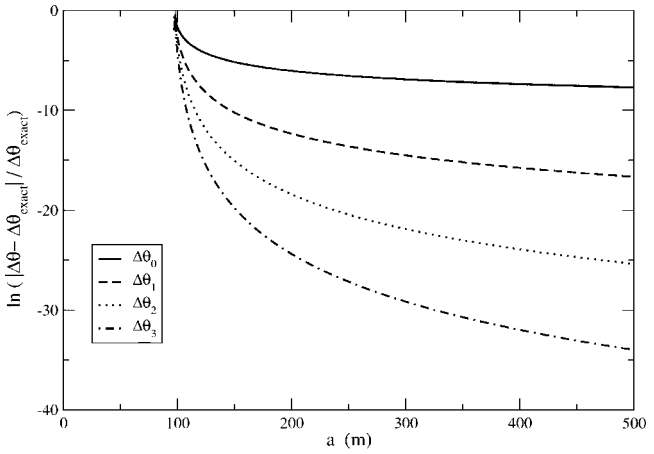

FIG. 5. Logarithmic plot of the error over the precession of the perihelion as a function a. The different lines correspond to the different orders to which our approximation has been applied.

and treat each integral in the way indicated above. However, we do not deem it necessary to discuss such calculation here because it does not add anything new to what was already done.

We can apply present method to anharmonic oscillators with more than two terms in the potential-energy function; as an example consider

$$V(x) = a_2x^2 + a_3x^3 + a_4x^4. \tag{51}$$

If we write

$$Q(x) = E - V(x) = (x - x_-)(x_+ - x)(b_0 + b_1x + b_2x^2), \tag{52}$$

then we obtain

$$E = -x_+x_-[a_2 + a_3(x_+ + x_-) + a_4(x_+^2 + x_+x_- + x_-^2)], \tag{53}$$

$$b_0 = a_2 + a_3(x_+ + x_-) + a_4(x_+^2 + x_+x_- + x_-^2), \tag{54}$$

$$b_1 = a_3 + a_4(x_+ + x_-), \tag{55}$$

$$b_2 = a_4, \tag{56}$$

and

$$\Delta = \frac{b_0 + b_1x + b_2x^2 - \omega^2}{\omega^2}. \tag{57}$$

It follows from the PMS condition $I_1=0$ that

$$\omega_{PMS} = \frac{\sqrt{16b_0 + 8b_1(x_- + x_+) + 2b_2(3x_-^2 + 2x_+x_- + 3x_+^2)}}{4}. \tag{58}$$

The expression of the period to order zero is quite simple $T\approx\sqrt{2}I_0=\sqrt{2}\pi/\omega_{PMS}$ but the expression to second order $T\approx\sqrt{2}(I_0+I_2)$, although simple, is rather long to be shown here.

E. The simple pendulum

The method just outlined also applies to nonpolynomial potentials. As a representative example, we consider the simple pendulum, and without loss of generality we choose unit mass and length. We obtain particularly simple expressions if we expand the potential

$$V(\phi) = 1 - \cos\phi \tag{59}$$

in a Taylor series. For example, in the first approximation $V(\phi)\approx\phi^2/2$ is harmonic and the period is independent of the amplitude A. The resulting textbook expression for the period is valid for small values of A. In order to test the accuracy of the approximations we consider the exact expression for the period [12]

$$T = 2\int_0^\pi \frac{d\alpha}{\sqrt{1 - \sin^2\frac{A}{2}\sin^2\frac{\alpha}{2}}}. \tag{60}$$

If we keep one more term in the expansion $V(\phi)\approx\phi^2/2 - \phi^4/24$ we can apply all the results for the Duffing oscillator

with $\mu=-1/6$. The leading term of the series (18) yields in this case

$$T=\frac{4\sqrt{2}\pi}{\sqrt{8-A^2}}. \tag{61}$$

Taking into account terms of higher order of the series (18) does not improve results because the mayor source of error is the expansion of the potential only through fourth order.

The next approximation in the expansion of the potential is $V(\phi)\approx\phi^2/2-\phi^4/24+\phi^6/720$. The present method yields

$$\omega_{PMS}=\frac{\sqrt{6}(A^4-24A^2+192)}{48} \tag{62}$$

at first order of PMS. Keeping just the leading term of the series (3) we obtain

$$T=\frac{16\sqrt{3}\pi}{\sqrt{A^4-24A^2+192}}. \tag{63}$$

This expression gives better results than the preceding one, and in this case the term of second order improves the accuracy (the term of first order is zero because of the PMS). We thus obtain

$$T=\frac{\sqrt{3}\pi(253A^8-11904A^6+233280A^4-2211840A^2+8847360)}{15(A^4-24A^2+192)}. \tag{64}$$

We can proceed in this way, adding more terms to the expansion of the potential-energy function and to the expansion of the integral, and increase the accuracy of the expression for the period as much as necessary.

V. CONCLUSIONS

In this paper we propose a technique for solving a class of integrals with applications in theoretical physics. We provide a systematic general method for writing those integrals in a form convenient for exact calculation and for their expansion in series. We show that the resulting series are suitable for all values of the relevant physical parameters, from unstable equilibrium through the strong coupling, and for all the anharmonic oscillators considered here. We easily derive simple low-order analytical expressions that are more accurate than those in the literature, and we have shown that the rate of convergence of our series is greater than those proposed by other authors. In some cases the greater convergence rate observed is probably due to the fact that the PMS works better on the period than on the frequency.

The approach that we propose in this paper is closely related to what is known as variational perturbation method, which was extensively applied to quantum mechanics and other fields of theoretical physics [8–10] (see also additional references cited in [2]). In particular, the method of nonlinear mappings [8], which focuses only on the series itself, is more general that the standard variational perturbation theory. The nonlinear mappings can also be applied to the problems considered here. However, we have resorted to the technique outlined above because it has facilitated the analysis of the convergence of the series by focusing on properties of the integrals.

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